RMSC 4003
Statistical Modeling in Financial Markets
Tutorial 9: Itô's Lemma (Solution)

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In the following, W_t always denotes a standard Brownian motion.

1 Itô's Lemma

Definition 1.1. *A* ***diffusion process*** *is a stochastic process X_t satisfying*

$$X_t = X_0 + \mu t + \sigma W_t,$$

where W_t is a standard Brownian motion.

Written in stochastic differential equation (SDE) form:

$$dX_t = \mu dt + \sigma dW_t.$$

You can think of dX_t as $X_{t+\Delta t} - X_t$, where Δt is very small.

Definition 1.2. *An* ***Itô process*** *X_t is a stochastic process satisfying the following SDE:*

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t,$$

where $\mu(x, t)$ is the ***drift function*** *and $\sigma(x, t)$ is the* ***volatility (or diffusion) function****.*

The meaning of the above SDE is

$$X_t = X_0 + \int_0^t a(X_s, s)ds + \int_0^t b(X_s, s)dW_s.$$

$\int_0^t f(s, X_s)dW_s$ is a stochastic integral (to be defined formally in RMSC4005), where $f(s, x)$ is a deterministic function.

Definition 1.3. *A stochastic process $S(t)$ is said to be a* ***geometric Brownian motion*** *if it satisfies the following SDE:*

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

By Itô's Lemma, it is equivalent for $\log S(t)$ to satisfies

$$d \log S_t = (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t.$$

Theorem 1.1 (Itô's Lemma). *Let X_t be an Itô process given by*

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t.$$

Let $F(t, x) \in C^2$. (i.e. F is twice continuously differentiable). Then $Y_t := F(t, X_t)$ satisfies

$$dY_t = \frac{\partial F}{\partial t}(t, X_t)dt + \frac{\partial F}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t, X_t)(dX_t)^2. \tag{1}$$

To write in the form in lecture notes, note that

$$\begin{aligned} dY_t &= \frac{\partial F}{\partial t}(t, X_t)dt + \frac{\partial F}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t, X_t)(dX_t)^2 \\ &= \frac{\partial F}{\partial t}(t, X_t)dt + \frac{\partial F}{\partial x}(t, X_t)\Big[a(X_t, t)dt + b(X_t, t)dW_t\Big] + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t, X_t)b^2 dt \\ &= \left[\frac{\partial F}{\partial t}(t, X_t) + a\frac{\partial F}{\partial x}(t, X_t) + \frac{1}{2}b^2\frac{\partial^2 F}{\partial x^2}(t, X_t)\right]dt + b\frac{\partial F}{\partial x}(t, X_t)dW_t. \end{aligned} \tag{2}$$

Note that the arguments in a and b are suppressed to avoid complicating the notations.

Remark 1.1. $(dX_t)^2 = b^2 dt$ since $dtdt = 0$, $dtdW_t = dW_t dt = 0$ and $dW_t dW_t = dt$. "Box Algebra Multiplication Table":

$\cdot$	dt	dW_t
dt	0	0
dW_t	0	dt

In the solution, the a and b will always refer to the "a" and "b" in the Itô process.

2 Finding the SDE

Possible steps:

(1) Define $f(x, t)$ and find a suitable Itô process X_t.

(2) Apply Itô's Lemma to $f(X_t, t)$.

Example 2.1. (a) Find $d(e^{\sigma W_t - \frac{1}{2}\sigma^2 t})$, where σ is a constant.

(b) (12-13 Final Q2) Suppose $dr_t = \alpha dt + \beta dW_t$. Show that $M_t := \exp(r_t - (\alpha + \frac{\beta^2}{2})t)$ follows a geometric Brownian motion.

Solution. (a) Let $f(x, t) = e^{\sigma x - \frac{1}{2}\sigma^2 t}$. Then $f_t = -\frac{1}{2}\sigma^2 f$, $f_x = \sigma f$ and $f_{xx} = \sigma^2 f$. Let $X_t = W_t$. Then $a = 0$ and $b = 1$. By Itô's Lemma,

$$\begin{aligned} d(e^{\sigma W_t - \frac{1}{2}\sigma^2 t}) &= df(X_t, t) \\ &= \big(-\frac{1}{2}\sigma^2 e^{\sigma W_t - \frac{1}{2}\sigma^2 t} + \frac{1}{2}\sigma^2 e^{\sigma W_t - \frac{1}{2}\sigma^2 t}\big)dt + \sigma e^{\sigma W_t - \frac{1}{2}\sigma^2 t}dW_t \\ &= \sigma e^{\sigma W_t - \frac{1}{2}\sigma^2 t}dW_t. \end{aligned}$$

Alternatively, as said in the tutorial (1830 session), you can also define $X_t = \sigma W_t - \frac{1}{2}\sigma^2 t$. In this case, we have

$$dX_t = -\frac{1}{2}\sigma^2 dt + \sigma dW_t.$$

Therefore, $a = -\frac{1}{2}\sigma^2$ and $b = \sigma$. This time, we have to define $f(x,t) = e^x$. Then, $f_t = 0, f_x = e^x$ and $f_{xx} = e^x$. By Itô's Lemma,

$$\begin{aligned} d(e^{\sigma W_t - \frac{1}{2}\sigma^2 t}) &= df(X_t, t) \\ &= \big(-\frac{1}{2}\sigma^2 e^{\sigma W_t - \frac{1}{2}\sigma^2 t} + \frac{1}{2}\sigma^2 e^{\sigma W_t - \frac{1}{2}\sigma^2 t}\big) dt + \sigma e^{\sigma W_t - \frac{1}{2}\sigma^2 t} dW_t \\ &= \sigma e^{\sigma W_t - \frac{1}{2}\sigma^2 t} dW_t. \end{aligned}$$

Thus, you will get the same answer.

(b) Let $f(x,t) = \exp(x - (\alpha + \frac{\beta^2}{2})t)$. Then $f_t = -(\alpha + \frac{\beta^2}{2})f, f_x = f_{xx} = f$. Note that $a = \alpha$ and $b = \beta$. By Itô's Lemma,

$$\begin{aligned} dM_t &= df(r_t, t) \\ &= \big(-(\alpha + \frac{\beta^2}{2})M_t + \alpha M_t + \frac{1}{2}\beta M_t\big) dt + \beta M_t dW_t \\ &= \beta M_t dW_t. \end{aligned}$$

Hence, M_t follows a geometric Brownian motion.

3 Finding the Integral

Possible steps:

(1) Use integration by parts to guess the answer.

(2) Apply Itô's Lemma on the first term that comes out from integration by parts.

(3) Integrate both sides and rearrange the terms.

Theorem 3.1 (Integration by parts)**.** *Let F, G be differentiable on $[a,b]$ and let $f := F'$ and $g := G'$ be integrable. Then*

$$\int_a^b f(x)G(x)dx = F(x)G(x)\Big|_a^b - \int_a^b F(x)g(x)dx.$$

Sometimes, we also write

$$\int_a^b G(x)dF(x) = F(x)G(x)\Big|_a^b - \int_a^b F(x)dG(x).$$

In lecture notes, we know that

$$\int_0^t s dW_s = tW_t - \int_0^t W_s ds$$

and

$$\int_0^t sW_s dW_s = \frac{t}{2}W_t^2 - \frac{t^2}{4} - \frac{1}{2}\int_0^t W_s^2 ds.$$

Example 3.1 (See Supplementary Examples Q1)**.** Show that for any $m = 0, 1, 2, \ldots$,

$$\int_0^t W_s^m dW_s = \frac{W_t^{m+1}}{m+1} - \frac{m}{2}\int_0^t W_s^{m-1} ds.$$

Solution. For $m = 0$, LHS $= W_t =$ RHS. For $m \geq 1$, let $f(x,t) := x^{m+1}$. Then $f_t = 0, f_x = (m+1)x^m$ and $f_{xx} = (m+1)mx^{m-1}$. Let $X_t = W_t$. Then $a = 0$ and $b = 1$. By Itô's Lemma,

$$dW_t^{m+1} = df(X_t,t) = \frac{(m+1)m}{2}W_t^{m-1}dt + (m+1)W_t^m dW_t.$$

Integrating both sides (recall that $W_0 = 0$),

$$W_t^{m+1} = \frac{(m+1)m}{2}\int_0^t W_s^{m-1}ds + (m+1)\int_0^t W_s^m dW_s.$$

Rearranging the terms,

$$\int_0^t W_s^m dW_s = \frac{W_t^{m+1}}{m+1} - \frac{m}{2}\int_0^t W_s^{m-1}ds.$$

Example 3.2. Let W_t be a standard Brownian motion. Evaluate

$$\int_0^t e^{-s}dW_s.$$

Solution. Let $f(x,t) = e^{-t}x$. Then $f_t = -e^{-t}x, f_x = e^{-t}$ and $f_{xx} = 0$. Let $X_t = W_t$. Then $a = 0$ and $b = 1$. By Itô's Lemma,

$$d(e^{-t}W_t) = df(X_t,t) = -e^{-t}W_t dt + e^{-t}dW_t.$$

Integrating both sides (recall that $W_0 = 0$),

$$e^{-t}W_t = -\int_0^t e^{-s}W_s ds + \int_0^t e^{-s}dW_s.$$

Rearranging the terms,

$$\int_0^t e^{-s}dW_s = e^{-t}W_t + \int_0^t e^{-s}W_s ds.$$

Example 3.3 (12-13 Final Q2)**.** Prove or disprove the following:

$$\int_0^t W_s \exp(W_s^2)dW_s = \frac{1}{2}\exp(W_t^2) - \int_0^t W_s^2 \exp(W_s^2)ds.$$

Solution. Let $f(x,t) = e^{x^2}$. Then $f_t = 0, f_x = 2xe^{x^2}$ and $f_{xx} = 2e^{x^2} + 4x^2e^{x^2}$. Let $X_t = W_t$. Then $a = 0$ and $b = 1$. By Itô's Lemma,

$$d(e^{W_t^2}) = df(X_t,t) = \frac{1}{2}(2e^{W_t^2} + 4W_t^2e^{W_t^2})dt + 2W_te^{W_t^2}dW_t.$$

Integrating both sides (recall that $W_0 = 0$ and $e^0 = 1$),

$$e^{W_t^2} - 1 = \int_0^t e^{W_s^2} + 2W_s^2e^{W_s^2}ds + 2\int_0^t W_se^{W_s^2}dW_s.$$

Rearranging the terms,

$$\int_0^t W_se^{W_s^2}dW_s = \frac{1}{2}(e^{W_t^2} - 1) - \frac{1}{2}\int_0^t e^{W_s^2} + 2W_s^2e^{W_s^2}ds.$$

Therefore, the statement is false.

4 Solving an SDE

Possible steps:

(1) Apply Itô's Lemma to a transformation (e.g. log) of the process or to the process multiplied by an integrating factor.

(2) Integrate the SDE.

(3) Reverse the transformation/ multiply the inverse of the integrating factor.

Example 4.1. Suppose the stock price $S(t)$ follows the geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Let $f(x,t) = \log x$. Then $f_t = 0, f_x = \frac{1}{x}$ and $f_{xx} = -\frac{1}{x^2}$. Note that $a = \mu S_t$ and $b = \sigma S_t$. By Itô's Lemma,

$$\begin{aligned} d\log S_t &= df(S_t,t) \\ &= \left(\mu S_t \frac{1}{S_t} + \frac{1}{2}(\sigma S_t)^2(-\frac{1}{S_t^2})\right)dt + \sigma dW_t \\ &= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t. \end{aligned}$$

Integrating both sides,

$$\log S_t - \log S_0 = (\mu - \frac{\sigma^2}{2})t + \sigma W_t.$$

Rearranging the terms and exponentiating,

$$S_t = S_0 \exp\left\{(\mu - \frac{\sigma^2}{2})t + \sigma W_t\right\}.$$

Remark 4.1. S_t is lognormally distributed. Hence, $\mathrm{E}(S_t)$ and $\mathrm{Var}(S_t)$ can be found as in Tutorial 8.

Example 4.2. Suppose S_t satisfies the generalized geometric Brownian motion differential equation:

$$dS_t = \alpha(t)S_t dt + \sigma(t)S_t dW_t.$$

Show that $S_t = S_0 \exp\left\{\int_0^t \sigma(s)dW_s + \int_0^t (\alpha(s) - \frac{1}{2}\sigma^2(s))ds\right\}$.

Solution. The proof is essentially the same as above. Let $f(x,t) = \log x$. Then $f_t = 0, f_x = \frac{1}{x}$ and $f_{xx} = -\frac{1}{x^2}$. Note that $a = \alpha(t)S_t$ and $b = \sigma(t)S_t$. By Itô's Lemma,

$$\begin{aligned} d\log S_t &= df(S_t,t) \\ &= \left(\alpha(t)S_t \frac{1}{S_t} + \frac{1}{2}(\sigma(t)S_t)^2(-\frac{1}{S_t^2})\right)dt + \sigma(t)dW_t \\ &= \left(\alpha(t) - \frac{\sigma^2(t)}{2}\right)dt + \sigma(t)dW_t. \end{aligned}$$

Integrating both sides,

$$\log S_t - \log S_0 = \int_0^t \sigma(s)dW_s + \int_0^t (\alpha(s) - \frac{1}{2}\sigma^2(s))ds.$$

Rearranging the terms and exponentiating,

$$S_t = S_0 \exp\left\{ \int_0^t \sigma(s) dW_s + \int_0^t (\alpha(s) - \frac{1}{2}\sigma^2(s)) ds \right\}.$$

Example 4.3. (Ornstein-Uhlenbeck process, see Supplementary Examples Q5) Solve

$$dX_t = \mu X_t dt + \sigma dW_t,$$

where μ and σ are real constants.

Solution. Let $f(x,t) = e^{-\mu t}x$. ($e^{-\mu t}$ is the integrating factor). Then $f_t = -\mu e^{-\mu t}x, f_x = e^{-\mu t}$ and $f_{xx} = 0$. Note that $a = \mu X_t$ and $b = \sigma$. By Itô's Lemma,

$$\begin{aligned} d(e^{-\mu t}X_t) &= df(X_t, t) \\ &= (-\mu e^{-\mu t}X_t + \mu X_t(e^{-\mu t}))dt + \sigma e^{-\mu t}dW_t \\ &= \sigma e^{-\mu t}dW_t. \end{aligned}$$

Integrating both sides,

$$e^{-\mu t}X_t - X_0 = \sigma \int_0^t e^{-\mu s} dW_s.$$

Rearranging the terms

$$X_t = e^{\mu t}X_0 + \sigma \int_0^t e^{\mu(t-s)} dW_s.$$

Example 4.4. (Mean-reverting Ornstein-Uhlenbeck process) Solve

$$dX_t = (m - X_t)dt + \sigma dW_t,$$

where m and σ are real constants.

Solution. Let $f(x,t) = e^t x$. Then $f_t = e^t x, f_x = e^t$ and $f_{xx} = 0$. Note that $a = m - X_t$ and $b = \sigma$. By Itô's Lemma,

$$\begin{aligned} d(e^t X_t) &= df(X_t, t) \\ &= (e^t X_t + (m - X_t)e^t)dt + \sigma e^t dW_t \\ &= me^t dt + \sigma e^t dW_t. \end{aligned}$$

Integrating both sides,

$$e^t X_t - X_0 = m(e^t - 1) + \sigma \int_0^t e^s dW_s.$$

Rearranging the terms,

$$X_t = e^{-t}X_0 + m(1 - e^{-t}) + \sigma \int_0^t e^{s-t} dW_s.$$

Example 4.5. The Vasicek interest rate stochastic differential equation is

$$dr_t = (\alpha - \beta r_t)dt + \sigma dW_t,$$

where α, β and σ are positive constants. Find r_t.

Solution. Let $f(x,t) = e^{\beta t}x$. Then $f_t = \beta e^{\beta t}x, f_x = e^{\beta t}$ and $f_{xx} = 0$. Note that $a = \alpha - \beta r_t$ and $b = \sigma$. By Itô's Lemma,

$$\begin{aligned} d(e^{\beta t}r_t) &= df(r_t,t) \\ &= (\beta e^{\beta t}r_t + (\alpha - \beta r_t)e^{\beta t})dt + \sigma e^{\beta t}dW_t \\ &= \alpha e^{\beta t}dt + \sigma e^{\beta t}dW_t. \end{aligned}$$

Integrating both sides,

$$e^{\beta t}r_t - r_0 = \alpha \int_0^t e^{\beta s}ds + \sigma \int_0^t e^{\beta s}dW_s.$$

Rearranging the terms,

$$r_t = r_0 e^{-\beta t} + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t} \int_0^t e^{\beta s}dW_s.$$

Example 4.6. Consider a zero-coupon bond price that pays 1 at T with constant interest rate:

$$dP_t = P_t r dt.$$

Find P_t.

Solution. This is in fact just an separable ODE.

$$\begin{aligned} \frac{dP_t}{P_t} &= rdt \\ d\log P_t &= rdt \\ \log P_t - \log P_0 &= rt \\ P_t &= P_0 e^{rt} \end{aligned}$$

Since $P_T = 1$, we know $P_0 = e^{-rT}$. Hence, $P_t = e^{-r(T-t)}$.

5 Appendix: Integrating factor

This section introduces how we can come up with the integrating factor that appears in solving SDE. Let y be a function of x. The general form of a first-order linear ODE is

$$y' + p(x)y = g(x).$$

The way to solve it (find y) is to multiply both sides by a suitable factor called **integrating factor**.

$$\begin{aligned} y' + p(x)y &= g(x) \\ e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}p(x)y &= g(x)e^{\int p(x)dx} \\ \frac{d}{dx}e^{\int p(x)dx}y &= g(x)e^{\int p(x)dx} \\ e^{\int p(x)dx}y &= \int g(x)e^{\int p(x)dx}dx \\ y &= e^{-\int p(x)dx}\int g(x)e^{\int p(x)dx}dx \end{aligned}$$

Example 5.1. Solve $y' + 2xy = 0$.

Solution.

$$\begin{aligned}
y' + 2xy &= 0 \\
e^{x^2}y' + 2xe^{x^2}y &= 0 \\
\frac{d}{dx}(e^{x^2}y) &= 0 \\
e^{x^2}y &= C \\
y &= Ce^{-x^2} \\
y(0) &= Ce^{-0} \\
y &= y(0)e^{-x^2}.
\end{aligned}$$

Example 5.2. (2012-2013 Final Q2) Consider a deterministic function $U(t)$ satisfying

$$dU_t = a(b - U_t)dt.$$

Show that

$$U_t = e^{-at}U_0 + b(1 - e^{-at}).$$

Solution.

$$\begin{aligned}
dU(t) &= a(b - U(t))dt \\
\frac{dU(t)}{dt} + aU(t) &= ab \\
e^{at}\frac{dU(t)}{dt} + e^{at}aU(t) &= e^{at}ab \\
\frac{d}{dt}(e^{at}U(t)) &= e^{at}ab \\
e^{at}U(t) - U(0) &= ab\int_0^t e^{as}\, ds \\
&= b(e^{at} - 1) \\
U(t) &= e^{-at}U(0) + b(1 - e^{-at}).
\end{aligned}$$